this channel. The state $D_{5/2}$ is suggested for this resonance. If this assignment is correct the question arises as to where the first recurrence of this trajectory with spin $\frac{1}{2}$ is. From the slopes of the other trajectories and from the fact that there is no particle between 1190 and 1765 MeV with the quantum numbers of this resonance one would expect its trajectory to lie higher than Σ trajectory. But also below 1190 MeV there is no particle with the quantum numbers of this resonance. Thus we are faced with the alternative that the odd parity trajectory crosses the spin $\frac{1}{2}$ line in the positive energy region thus having the wrong region for the odd trajectory and having the wrong slope for the even trajectory. A wrong slope would give a negative width and would not correspond to a particle. This possibility is shown in Fig. 6.

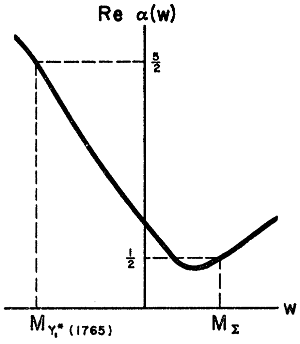

FIG. 6. A possible connection between the trajectories of Σ and 1765-MeV Y_1^*.

PHYSICAL REVIEW VOLUME 135, NUMBER 1B 13 JULY 1964

Dispersion Approach to Two-Body Weak Decays*

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(Received 6 February 1964)

A dispersion theoretic approach to two-body weak decays is discussed in which the masses of the particles are regarded as constants. In this approach, an analyticity assumption is introduced for the invariant decay amplitudes which are defined off the energy-momentum shell. These amplitudes are invariant functions of three invariant variables. The singularities and the dispersion relations for these amplitudes are very similar to those assumed by Mandelstam in the case of scattering. As examples, the pionic decays of hyperons and the leptonic decays of pions and kaons are discussed in detail in the first order with respect to the weak Hamiltonian. It is assumed in the present approach that there exists a weak Hamiltonian which is localized in the sense of the present local field theory and can be treated as a small perturbation. In the case of the former example, it is shown that there are three kinds of pole terms corresponding to the above three invariant variables and that these pole terms are identical with those assumed in the pole approximation due to Feldman, Matthews, and Salam. The invariant decay amplitude in the case of the latter example becomes a constant in the present approach, if the electromagnetic correction is ignored. This is to be contrasted with various dispersion relations proposed in the conventional approach in which the mass of the pion is regarded as the variable. The dispersion theoretic version of the usual $V-A$ theory of the weak interaction is then constructed, in which the invariant decay amplitude (being a constant) is regarded essentially as the weak coupling constant. The experimental data concerning these leptonic decays indicate that the weak coupling constant defined this way is independent of not only whether the charged lepton is the electron or the μ meson, but also whether the decaying particle is the pion or the kaon.

I. INTRODUCTION

IN the dispersion theoretic approach to scattering, one usually assumes analyticity of invariant scattering amplitudes with respect to the invariant combinations of the particle four-momenta. These four-momenta are subject to the over-all energy-momentum conservation and all remain on the respective mass shells. One then finds two independent invariant variables. If one assumes analyticity with respect to both of these variables, one obtains double dispersion relations for the invariant scattering amplitudes. This was done first by Mandelstam.[1]

Suppose one applies the same consideration to decay of a particle with mass M into two particles with masses M' and m, respectively. It is straightforward to define invariant decay amplitudes. However, one finds no invariant variables, if all the particle four-momenta remain on the respective mass shells and satisfy the over-all energy-momentum conservation. To see this, let p, p', and q be the four-momenta[2] of the particles with masses M, M', and m, respectively. The conditions that these momenta are on the respective mass shells and satisfy the over-all energy-momentum conservation

* Work supported by the National Science Foundation.

[1] S. Mandelstam, Phys. Rev. **115**, 1741 (1959).

[2] Our notation of the four-momentum p is such that the space components are those of the three-momentum $\mathbf{p}$, and the fourth component is ip_0, where p_0 is the relativistic energy of this particle.

are expressed as

$$p^2=-M^2,\quad p'^2=-M'^2,\quad q^2=-m^2, \tag{1}$$

$$p=p'+q. \tag{2}$$

If one regards two momenta out of these three as independent, one finds three independent invariant combinations of these momenta. However, because of (1) and (2), all these invariants are merely the three-particle masses. Therefore, in the case of two-body decays, one cannot introduce an analyticity assumption of the kind which is usually assumed in the case of scattering.

One can still introduce analyticity assumptions for two-body decays, if one regards some of the particle masses as variables and considers invariant decay amplitudes as functions of these variable masses. This is what has been done so far in the applications of dispersion relations to some of the two-body weak decays. In particular, the leptonic decays of pions were analyzed by Goldberger and Treiman[3] and subsequently by many others,[4] using the pion mass as the variable. The pionic decays of hyperons were also discussed by McCliment and Nishijima,[5] using the mass of the decaying hyperon as the variable.

The purpose of the present paper is to investigate the alternative way of introducing an analyticity assumption, in which the masses of the particles are regarded as constants. In this approach, all the particle four-momenta are strictly on the mass shells given by (1). The only way to introduce the variables is then to consider the decay matrix element off the energy-momentum shell, that is the matrix element of the original Hamiltonian responsible for the decay with respect to the same kinds of particles, the four-momenta of which do not have to satisfy the over-all energy-momentum conservation (2). The invariant decay amplitudes then become invariant functions of three invariant variables, which can be chosen as s, t, and u, given by

$$s=-(p'+q)^2,\quad t=-(p-p')^2,\quad u=-(p-q)^2. \tag{3}$$

The number of the independent variables is reduced to two, if one requires that

$$s+t+u=M^2+m^2+M'^2, \tag{4}$$

which includes the physical region in these variables

$$s=M^2,\quad t=m^2,\quad u=M'^2. \tag{5}$$

The fundamental postulate in the present approach is the assumption that the invariant decay amplitudes defined off the energy-momentum shell are analytic in s, t, and u except for the cuts and poles which appear along the real axes of these variables and satisfy the simplest dispersion relations consistent with the above singularities.

We add two remarks to the above discussion. First, without the condition (2), one finds generally more invariant decay amplitudes than one has with it. However, this does not cause any difficulty as long as the invariant decay amplitudes are defined in such a way that they tend to the invariant decay amplitudes on the energy-momentum shell when the physical region is approached. The details are explained in Secs. II and III.

Secondly, the condition (4) is equivalent to

$$k+p=p'+q,\quad k^2=0. \tag{6}$$

One can see this, by observing that $s+t+u-M^2-M^2-M'^2$ is equal to $-(\mathrm{p}-\mathrm{p}'-q)^2$ because of (1) and (3). Because of this equivalence, one may have the following picture of the present approach. One postulates a fictitious massless particle (also neutral and spinless) which is annihilated in the two-body decays in such a way that the corresponding decay matrix elements become the physical decay matrix elements in the limit when this fictitious particle carries no energy and momentum. However, the present author is not inclined to take this picture seriously. The basic motivation for requiring the condition (4), which is equivalent to (6), is to minimize the amount of unphysical continuation without excluding the physical region (5). For this purpose, the present author finds no other way than requiring (4) or (6). For example, one could reduce the number of the independent variables by putting some of s, t, and u equal to their physical values (5). However, there is no *a priori* criterion for which variables ought to be fixed. Besides, one might in this way overlook some of the analytic structure of the decay amplitude which could otherwise be very useful.

In Secs. II and III, we explain in detail how to define the invariant decay amplitudes off the energy-momentum shell and how to locate the singularities with respect to the invariant variables. We discuss, as examples, the pionic decays of hyperons in Sec. II and the leptonic decays of pions and kaons in Sec. III. We assume in these sections that there exists a weak Hamiltonian $H_W(x)$, which is localized in the sense of the present local field theory, and can be treated as a small perturbation, and is responsible for these weak decays in the first order. However, one does not have to know the explicit form of $H_W(x)$. The analyses in Secs. II and III refer to the first-order matrix element of this weak Hamiltonian, ignoring the electromagnetic correction. However, all the strong interactions are included. If some of the higher order matrix elements of the weak Hamiltonian and/or of the electromagnetic correction are included, additional singularities have to be added to the invariant decay amplitudes, but these higher order corrections do not cause any basic

[3] M. L. Goldberger and S. B. Treiman, Phys. Rev. **110**, 1178 (1958); **111**, 354 (1958).
[4] Some of the recent works are M. Ida, Phys. Rev. **132**, 401 (1963) and K. Nishijima, Phys. Rev. **133**, B1092 (1964), which contain references to all previous work.
[5] E. R. McCliment and K. Nishijima, Phys. Rev. **128**, 1970 (1962).

difficulties in formulating the present approach. However, the existence of a localized weak Hamiltonian appears to be a crucial assumption in this approach, because the present author does not know how to define the decay matrix element off the energy-momentum shell when there is no localized weak Hamiltonian.

In the final section, we summarize the analyses in Secs. II and III, with particular emphasis on comparison between the present approach and the conventional approach.[3–5]

II. PIONIC DECAYS OF HYPERONS

We assume that a localized weak Hamiltonian $H_W(x)$ is responsible for the pionic decays of hyperon in the first order. We use those notations of masses and four-momenta which are introduced in Sec. I, except that the particle with mass m is identified in this section as the pion. In this and the following sections, all the state vectors and the operators are in the exact Heisenberg picture in which all the strong interactions, $H_S(x)$, are included. In the first order of $H_W(x)$, the over-all decay matrix element S_{fi} is given by

$$S_{fi}=-i\langle p',q|\int H_W(x)dx|p\rangle$$
$$=(2\pi)^4\delta(p-p'-q)(-i)\langle p',q|H_W(0)|p\rangle, \quad (7)$$

where the integral is the over-all space-time integral.[6] According to the covariance argument, the matrix element in the last expression in (7) can be written as

$$-i(2q_0)^{1/2}\langle p',q|H_W(0)|p\rangle=\{\bar{u}(p')[F\gamma_5+F']u(p)\}, \quad (8)$$

where $u(p)$ and $u(p')$ are free Dirac spinors, normalized as $u^\dagger u=1$; $\bar{u}(p')$ stands for $u^\dagger(p')\gamma_4$; and F and F' are invariant functions of the invariant variables. As is explained in Sec. I, the invariant variables are all the masses of the particles involved, and, therefore, F and F' are constants as long as the masses are regarded as constants. The parity-conserving part F accounts for the P-wave amplitude in the final state, while the parity-violating part F' is essentially the S wave final amplitude.

The decay matrix element off the energy-momentum shell is defined as the same as the matrix element in the last expression in (7), except that the three momenta p, p', and q satisfy (4) or equivalently (6), instead of (2). According to the covariance argument, this matrix element can be written as

$$-i(2q_0)^{1/2}\langle p',q|H_W(0)|p\rangle$$
$$=(\bar{u}(p')\{F(s,t,u)\gamma_5+F'(s,t,u)$$
$$+i\gamma\cdot k[G(s,t,u)\gamma_5+G'(s,t,u)]\}u(p)), \quad (9)$$

where s, t, and u are defined by (3) and all the F's and G's are invariant functions of s, t, and u. The expression (9) contains four invariant decay amplitudes compared with two in the expression (8). However, the F's and G's are defined in (9) in such a way that the F's in (9) tend to the F's in (8) as $k\to 0$. In other words, the following relations hold:

$$F=F(s=M^2,t=m^2,u=M'^2),$$
$$F'=F'(s=M^2,t=m^2,u=M'^2). \quad (10)$$

Therefore, one does not have to consider the G's in (9). The fundamental postulate in the present approach is that the F's in (9) are analytic in s, t, and u except for poles and cuts along the real axes of these variables.

We assume that the above singularities can be located according to the usual heuristic argument. Thus, one first eliminates[7] the pion in the decay matrix element off the energy-momentum shell. Ignoring the electromagnetic correction, one obtains

$$-i(2q_0)^{1/2}\langle p',q|H_W(0)|p\rangle$$
$$=-i\langle p'|\frac{\delta H_W(0)}{\delta\phi^\dagger(0)}|p\rangle-\langle p'|\int dxT\left\{\frac{\delta H_S(x)}{\delta\phi^\dagger(x)},H_W(0)\right\}$$
$$\times e^{-iqx}|p\rangle, \quad (11)$$

where $\phi(x)$ is the pion field operator and T stands for the time-ordered product. This time-ordered product can be written as

$$T\left\{\frac{\delta H_S(x)}{\delta\phi^\dagger(x)},H_W(0)\right\}$$
$$=H_W(0)\frac{\delta H_S(x)}{\delta\phi^\dagger(x)}+\eta(t)\left[\frac{\delta H_S(x)}{\delta\phi^\dagger(x)},H_W(0)\right], \quad (12)$$

where $\eta(t)$ is unity when $t>0$ and zero when $t<0$, and the last bracket is the usual commutator. The first term in (12) does not contribute in (11) because

$$\langle p'|\int dxH_W(0)\frac{\delta H_S(x)}{\delta\phi^\dagger(x)}e^{-iqx}|p\rangle$$
$$=\sum_n(2\pi)^4\delta(p-q-p_n)\langle p'|H_W(0)|n\rangle$$
$$\times\langle n|\frac{\delta H_S(0)}{\delta\phi^\dagger(0)}|p\rangle=0, \quad (13)$$

where the sum over n is over the complete set of the eigenstate $|n\rangle$ of the total strong Hamiltonian [thus excluding $H_W(x)$] and p_n in the corresponding energy-momentum eigenvalue. The last equality is due to the stability of hyperons against all the strong interactions. If one introduces the same complete set expansion also to the second term in (12), one can perform the space-

[6] The units $\hbar=c=1$ are used throughout this paper.

[7] F. E. Low, Phys. Rev. **97**, 1392 (1955).

time integral in (11). The result is

$$-i(2q_0)^{1/2}\langle p',q|H_W(0)|p\rangle=-i\langle p'|\frac{\delta H_W(0)}{\delta\phi^\dagger(0)}|p\rangle-i(2\pi)^3\sum_n\left\{\langle p'|\frac{\delta H_S(0)}{\delta\phi^\dagger(0)}|n\rangle\langle n|H_W(0)|p\rangle\frac{\delta(\mathbf{p}_n-\mathbf{p}'-\mathbf{q})}{p_0'+q_0-p_{n0}+i\epsilon}\right.$$
$$\left.+\langle p'|H_W(0)|n\rangle\langle n|\frac{\delta H_S(0)}{\delta\phi^\dagger(0)}|p\rangle\frac{\delta(\mathbf{p}_n-\mathbf{p}+\mathbf{q})}{p_0-q_0-p_{n0}-i\epsilon}\right\},\quad(14)$$

where ϵ is an infinitesimal positive real number.

According to the usual heuristic argument, the singularities of the invariant decay amplitudes are entirely due to the energy denominators in (14). Thus, one can locate all the singularities in s by looking at the mass spectrum of the intermediate state $|n\rangle$ which contributes to the first sum in (14), while the same consideration regarding the second sum in (14) enables one to locate all the singularities in u.

However, the expression (14) does not allow one to locate the singularities in t. For this purpose, one eliminates[7] the nucleon (or Λ in the Ξ decay) in the decay matrix element off the energy-momentum shell. The analogous procedure leads one to the expression

$$-i\langle p',q|H_W(0)|p\rangle=-i\bar{u}(p')\langle q|\frac{\delta H_W(0)}{\delta\bar{\psi}(0)}|p\rangle-i(2\pi)^3\sum_n\left\{\bar{u}(p')\langle q|\frac{\delta H_S(0)}{\delta\bar{\psi}(0)}|n\rangle\langle n|H_W(0)|p\rangle\frac{\delta(\mathbf{p}_n-\mathbf{p}'-\mathbf{q})}{p_0'+q_0-p_{n0}+i\epsilon}\right.$$
$$\left.+\langle q|H_W(0)|n\rangle\bar{u}(p')\langle n|\frac{\delta H_S(0)}{\delta\bar{\psi}(0)}|p\rangle\frac{\delta(\mathbf{p}_n-\mathbf{p}+\mathbf{p}')}{p_0-p_0'-p_{n0}-i\epsilon}\right\},\quad(15)$$

where $\bar{\psi}(x)=\psi^\dagger(x)\gamma_4$ and $\psi(x)$ is the nucleon (or Λ in the Ξ decay) field operator and other notations are the same as those in (14). The second sum in the expression (15) enables one to locate all the singularities with respect to t.

According to the above heuristic argument, poles are due to the single-particle intermediate states which contribute to the sum in (14) and (15). One thus finds that there can be poles in s because of the intermediate nucleon (or Σ in the Ξ decay), and in u due to the intermediate hyperon, and also in t by virtue of the intermediate kaon. One also finds three cuts given by $s\geq s_1$, $u\geq u_1$, and $t\geq t_1$, where s_1, u_1, and t_1 are the lowest total mass squared of the intermediate states which contribute to the respective sums in (14) and (15) and involve at least two particles.

The residues of the poles depend upon various strong and weak form factors. For example, the residue of the s pole includes the invariant form factors defined by

$$\langle p'|\delta H_S(0)/\delta\phi^\dagger(0)|p''\rangle=ig(z)[\bar{u}(p')\gamma_5u(p'')],\quad(16)$$

$$\langle p''|H_W(0)|p\rangle=\{\bar{u}(p'')[a(z)+a'(z)\gamma_5]u(p)\},\quad(17)$$

with $z\equiv-(p'-p'')^2$ in (16) and $z=-(p''-p)^2$ in (17). The invariant form factors, $g(z)$, $a(z)$, and $a'(z)$ are real, if $H_S(x)$ and $H_W(x)$ are time-reversal invariant. With (16) and (17), one can carry out the sum over the spin of the intermediate nucleon (or Σ in the Ξ decay) implied in the sum over n in (14). If one introduces covariant notation in the resulting expression of the term in (14) which gives rise to the s pole, one obtains

$$\begin{aligned}-i(2q_0)^{1/2}&\langle p',q|H_W(0)|p\rangle\\&=[g(m^2)/(s-M_0^2)](\bar{u}(p')\{(M_0+M)a(0)\gamma_5\\&\quad+(M_0-M)a'(0)+i\gamma\cdot k[a(0)\gamma_5+a'(0)]\}u(p))\\&\quad+\cdots,\end{aligned}\quad(18)$$

where M_0 is the mass of the intermediate nucleon (or Σ in the Ξ decay) and dots stand for the terms which do not have the pole in question. By comparing (18) with (9), one finds

$$\begin{aligned}F(s,t,u)&=[(M_0+M)g(m^2)a(0)/(s-M_0^2)]+\cdots,\\F'(s,t,u)&=[(M_0-M)g(m^2)a'(0)/(s-M_0^2)]+\cdots,\end{aligned}\quad(19)$$

where dots stand for the terms without the s pole.

If one applies the same consideration to the term in (14) which gives rise to the u pole, one obtains

$$\begin{aligned}F(s,t,u)&=[(M_0+M')g(m^2)a(0)/(u-M_0^2)]+\cdots,\\F'(s,t,u)&=[(M_0-M')g(m^2)a'(0)/(u-M_0^2)]+\cdots,\end{aligned}\quad(20)$$

where dots stand for the terms which do not have the u pole. In (20), M_0 is the mass of the intermediate hyperon responsible to the u pole and the definitions of $g(m^2)$, $a(0)$, and $a'(0)$ are the same as those in (16) and (17).

Concerning the term in (15) which gives rise to the t pole, one first observes the following identity relation which is valid when $p=p'+q'$.

$$\bar{u}(p')\langle q'|\frac{\delta H_S(0)}{\delta\bar{\psi}(0)}|p\rangle=\frac{1}{(2q_0')^{1/2}}\langle p'|\frac{\delta H_S(0)}{\delta\phi_K^\dagger(0)}|p\rangle,\quad(21)$$

where $\phi_K(x)$ is the kaon field operator. The identity relation (21) can be proved easily by eliminating[7] the kaon on the left-hand side and the nucleon (or Λ in the Ξ decay) on the right-hand side in (21). Therefore, one introduces the invariant form factors by putting

$$\bar{u}(p')\langle q'|\frac{\delta H_S(0)}{\delta\bar{\psi}(0)}|p\rangle=\frac{1}{(2q_0')^{1/2}}ig_K(z)[\bar{u}(p')\gamma_5 u(p)], \quad (22)$$

$$\langle q|H_W(0)|q'\rangle=\frac{1}{(4q_0q_0')^{1/2}}a_K(z), \quad (23)$$

with $z=-(p'-p)^2$ in (22) and $z=-(q-q')^2$ in (23). Equation (22) is correct at least for the purpose of finding the residue of the t pole. In (22) and (23), $g_K(z)$ and $a_K(z)$ are real if $H_S(x)$ and $H_W(x)$ are time-reversal invariant. Upon substituting (22) and (23) into the term in question in (15), one finds, after introducing covariant notation, that

$$F(s,t,u)=[g_K(m_K^2)a_K(0)/(t-m_K^2)]+\cdots, \quad (24)$$
$$F'(s,t,u)=\cdots,$$

where m_K is the kaon mass and dots stand for the terms which do not have the t pole. The absence of the t pole in $F'(s,t,u)$ is simply due to (23) in which the parity-violating part of $H_W(x)$ has no contribution to the matrix element in (23).

In addition, one obtains (19) also from the first sum in (15). This can be shown with the help of the identity relation (21).

All the preceding analyses which begin with Eq. (11) are valid, for any choice of the independent variables. Thus, one finds the same poles with the same residues and the same cuts, regardless of whether the condition (4) or equivalently (6) is required. However, the situation changes if one chooses the masses of the particles as variables, with the over-all energy-momentum conservation (2) required. In this case, s in (19) is M^2 and u in (20) is M'^2. Since M and M' appear also in the numerators in (19) and (20), one has to conclude that $F'(s,t,u)$ has neither an s pole nor a u pole, unless the invariant form factors $g(z)$ and $a'(z)$ become singular at these points. However, it is contrary to the usual heuristic argument to assume such singular behavior in the invariant form factors. It is a simple matter to check that the pole terms in (19), (20), and (24) are those which are assumed in the pole approximation due to Feldman, Matthews, and Salam,[8] if our g's and a's are identified with their vertex constants.

One notices that $g(m^2)$ in (19) and (20) is what one calls the coupling constant of the pion with baryons in the dispersion relations for the pion-baryon scattering amplitudes. The same remark applies also to $g_K(m_K^2)$ in (24). Therefore, it is very natural to regard the a's in (19), (20), and (24) as weak coupling constants of some type. The usefulness of defining the weak coupling constants this way is discussed in Secs. III and IV.

There is a simple graphical way to enumerate the singularities of the invariant decay amplitudes. First, one introduces the fictitious particle mentioned in Sec. I to the intitial state, in addition to the decaying particle. The decay diagram then consists of four particles, rather than the original three. In order to locate the singularities with respect to s, t, aud u, one proceeds exactly the same way as one usually does in the case of scattering, except that the vertices which the fictitious particle end up with are always the weak vertices, and the other vertices are to be regarded as the strong ones. This rule follows directly from the expressions (14) and (15).

One of the fundamental postulates in the present approach is that the invariant decay amplitudes satisfy the simplest dispersion relations consistent with the above singularities. Therefore, we assume that $F(s,t,u)$ and $F'(s,t,u)$ satisfy the following double dispersion relations:

$$F(s,t,u)=\frac{R_s}{s-s_0}+\frac{R_t}{t-t_0}+\frac{R_u}{u-u_0}+\frac{1}{\pi^2}\left[\int\int_{s_1t_1}^{\infty}\frac{\rho_{12}(s',t')ds'dt'}{(s'-s)(t'-t)}+\int\int_{t_1u_1}^{\infty}\frac{\rho_{23}(t',u')dt'du'}{(t'-t)(u'-u)}+\int\int_{u_1s_1}^{\infty}\frac{\rho_{31}(u',s')du'ds'}{(u'-u)(s'-s)}\right], \quad (25)$$

in terms of the obvious notation and a similar expression for $F'(s,t,u)$. In (25), $F(s,t,u)$ is assumed to have one pole in each of s, t, and u, since the exact number of poles is completely irrelevant in the following discussion.

One finds from (25) the limit of $F(s,t,u)$ when one of the variables become infinite while the other remains finite. For example, when $s\to\infty$ with t fixed, one obtains from (25) that

$$\lim_{s\to\infty}[F(s,t,u)]_{t\,\mathrm{fixed}}=R_t/(t-t_0), \quad (26)$$

provided that the double integrals in (25) converge individually for all values of s. One obtains similar expressions as (26) when other limits are considered. When all of s, t, and u become infinite, one obtains also from (25) that

$$\lim_{s,t,u\to\infty}F(s,t,u)=0. \quad (27)$$

The single dispersion relations for $F(s,t,u)$ can also be inferred from (25). For example, when $F(s,t,u)$ is regarded as a function of s with t fixed, (26) implies that $F(s,t,u)$ approaches a real finite number at infinity

[8] G. Feldman, P. T. Matthews, and A. Salam, Phys. Rev. **121**, 302 (1961).

in the s plane with t fixed. Therefore, $F(s,t,u)$ satisfies

$$F(s,t,u)=\frac{R_s}{s-s_0}+\frac{R_t}{t-t_0}+\frac{R_u}{u-u_0}+\frac{1}{\pi}\left[\int_{s_1}^{\infty}\frac{\rho_1(s',t)}{s'-s}ds'+\int_{u_1}^{\infty}\frac{\rho_3(u',t)}{u'-u}du'\right], \quad (28)$$

in which the limit (26) is used.[9] In fact, one obtains (26) also from (28) in the limit when s approaches infinity, provided that the integrals in (28) converge individually for all s. One can show[9] that the discontinuities in (25) and (28) are related, for example, by

$$\rho_1(s',t)=\frac{1}{\pi}\left[\int_{t_1}^{\infty}\frac{\rho_{12}(s',t')}{t'-t}dt'-\int_{u_1}^{\infty}\frac{\rho_{31}(u',s')}{t'-t}du'\right], \quad (29)$$

where t' in the last integral stands for $M^2+M'^2+m^2-u'-s'$.

III. LEPTONIC DECAYS OF PIONS AND KAONS

The leptonic decays of negative pions and kaons are considered in this section. The notation which is introduced in Secs. I and II is used, except that now the particle with mass M' is identified as the neutrino. It is assumed that the neutrino mass is zero ($M'=0$) and that the neutrino has the negative helicity. The question of whether or not the neutrino is the same in these leptonic decays is irrelevant. What is important to the following discussion is that a local weak Hamiltonian $H_W(x)$ is responsible for these leptonic decays in the first order, and the leptons participate only in the weak interaction.

The over-all decay matrix element is given also by (7) in the first order of $H_W(x)$. The matrix element in the last expression in (7) can, in this case, be written as

$$-i(2p_0)^{1/2}\langle p',q|H_W(0)|p\rangle=[\bar{u}(q)F(1+\gamma_5)u(-p')], \quad (30)$$

compared with (8) in the previous case. The difference between (8) and (30) is simply due to the requirement that the neutrino has the negative helicity. The constant F is dimensionless and is real if $H_W(x)$ is time-reversal invariant. The decay matrix element off the energy-momentum shell is defined in exactly the same way as in Sec. II. This matrix element can be written also as (9), except that $G(s,t,u)$ and $G'(s,t,u)$ are the same in this case. The relation (10) holds also in this case.

In order to locate the singularities of $F(s,t,u)$, it is the most convenient to eliminate[7] the leptons in the above decay matrix element. Because of the fact that the leptons participate only in the weak interaction, one obtains, when the electromagnetic correction is ignored,

$$-i(2p_0)^{1/2}\langle p',q|H_W(0)|p\rangle=-i(2p_0)^{1/2}[\bar{u}(q)\langle 0|\delta^2H_W(0)/\delta\bar{\Psi}(0)\delta\psi_\nu(0)|p\rangle u(-p')], \quad (31)$$

where $\psi_\nu(x)$ is the neutrino field operator and $\Psi(x)$ is that of the charged lepton. Since the matrix element on the right-hand side of (31) depends only on the four-momentum p, the covariance argument leads to

$$-i(2p_0)^{1/2}\langle 0|\delta^2H_W(0)/\delta\bar{\Psi}(0)\delta\psi_\nu(0)|p\rangle=F_1(1+\gamma_5)+F_1'(1-\gamma_5)-(i\gamma\cdot p/M)\{F_2(1+\gamma_5)+F_2'(1-\gamma_5)\}, \quad (32)$$

where M is inserted in the last term to make all the F's dimensionless. In (32), all the F's are invariant functions of p^2 and, therefore, are constants as long as M is regarded as a constant. From (31) and (32), one obtains

$$-i(2p_0)^{1/2}\langle p',q|H_W(0)|p\rangle=\{\bar{u}(q)[F_1+(m/M)F_2+(i\gamma\cdot k/M)F_2]\times(1+\gamma_5)u(-p')\}, \quad (33)$$

where M' is put equal to zero.

By comparing (33) with (9), one finds that $F(s,t,u)$ is a constant and, thus, equal to F in (30):

$$F(s,t,u)=F_1+(m/M)F_2=F. \quad (34)$$

Therefore, according to the present approach, the invariant decay amplitude $F(s,t,u)$ satisfies no (non-trivial) dispersion relation, in contrast with various dispersion relations[3–5] proposed in the conventional approach. The basic reasons for the result (34) are that the leptons participate only in the weak interaction and that the electromagnetic correction is ignored.

In the dispersion theoretic approach to scattering, various constants in the dispersion relations (the residues in the pole terms and the subtraction constants) are regarded as coupling constants of some type. Similarly, in the dispersion theoretic approach to the weak interaction, some constants in the dispersion relations for the invariant decay amplitudes ought to be identified as weak coupling constants of some kind. Since the invariant decay amplitudes themselves are constants in the leptonic decays of pions and kaons, it is quite natural in the present approach that one regards these invariant amplitudes as essentially the weak coupling constants.

One here observes that the usual V–A theory of the weak interaction predicts that there is in (32) only the term with F_2 and that this constant F_2 is independent of whether the charged lepton is the electron or the μ meson. One can construct a dispersion theoretic version of the usual V–A theory, by defining the weak coupling constant C by

$$F=(m/M)C, \quad (35)$$

[9] Those who are interested in seeing the details of this arugment are referred to M. Sugawara and A. Kanazawa, Phys. Rev. **123**, 1895 (1961).

and requiring that C is independent of whether the charged lepton is the electron or the μ meson. This version of the usual $V-A$ theory makes the same prediction as the latter regarding the following ratios:

$$w(\pi\to e+\nu)/w(\pi\to\mu+\nu)=1.28\times10^{-4},$$
$$w(K\to e+\nu)/w(K\to\mu+\nu)=0.257\times10^{-4}. \quad (36)$$

However, the above version is more general than the usual $V-A$ theory because the former does not necessarily require the usual form of $H_W(x)$ but is valid under a wider class of $H_W(x)$.

According to the usual $V-A$ theory, it would not be a meaningful question to ask if the weak coupling constant C, defined in (35), is also independent of whether the decaying particle is the pion or the kaon. However, this question is of direct significance according to the present approach, because this concerns directly the universality of the weak interaction. We have, therefore, determined the empirical values of the two weak coupling constants C_π and C_K, defined in exactly the same way in the leptonic decays of these two particles, respectively. Using the available data concerning the μ-mesonic decay modes, we have found that

$$C_K/C_\pi=0.97\,, \quad (37)$$
$$C_\pi=1.50\times10^{-7}. \quad (38)$$

Presumably the most ambiguous experimental figure in determining the above ratio is that of the branching ratio for $K\to\mu+\nu$. The value in (37) is based upon 64% determined by Roe *et al.*[10] All the previous values[10] of this branching ratio are consistently lower than 64% and make the figure in (37) smaller roughly by 5%. According to the quoted uncertainties of the experimental data which are necessary to determine the ratio C_K/C_π, the figure in (37) is uncertain by not more than roughly 2%. This leaves a discrepancy from unity of this ratio of about 1%.

Besides these experimental uncertainties, there are some theoretical corrections. Presumably the most important of all is the electromagnetic correction. With the electromagnetic interaction included on the left-hand side of (31), one obtains another term on the right-hand side of (31), upon eliminating[7] the leptons, due to the fact that the charged lepton participates also in the electromagnetic interaction. This additional term gives rise to some analytic structure of the invariant decay amplitude $F(s,t,u)$. Therefore, it would no longer be adequate to regard the values of $F(s,t,u)$ in its physical region as essentially the weak coupling constant. It would still be meaningful to define the weak coupling constant referring only to the term which appears on the right-hand side of (31). In other words, the weak coupling constant would have to be determined not directly from the observed decay rate but rather after the electromagnetic correction is subtracted.

In spite of these uncertainties which are mentioned above, it is very tempting to investigate whether the weak interaction is universal, in the sense that the weak coupling constants defined as outlined above become the same in the other weak decays also. For example, we have defined, in the case of the pionic decays of hyperons, the various constant a's which appear in the pole terms (19), (20), and (24) of Sec. II. These a's are not yet dimensionless and, therefore, cannot directly be compared with the c's in this section. The purpose of a subsequent paper[11] is to show that the available data actually suggest that these a's, after having been made dimensionless, are all numerically the same as that in (38).

IV. SUMMARY AND DISCUSSION

We have discussed in the previous sections how one can introduce an analyticity assumption for the two-body weak decays without regarding the masses of the particles as variables. For this purpose, one first defines the decay matrix element off the energy-momentum shell. This introduces the invariant decay amplitudes off the energy-momentum shell which are the invariant functions of the invariant variables s, t, and u defined by (3). One then assumes as the fundamental postulate in the present approach that these invariant decay amplitudes are analytic in s, t, and u except for the poles and cuts which occur along the real axes of these variables and behave in possibly the simplest way at infinity with respect to these variables.

The above singularities can be located more or less in the same way as they are in the case of scattering. This is because one is in the present approach off the energy-momentum shell and, thus, one has an extra energy momentum which can formally be attributed to the fictitious particle present initially, in addition to the decaying particle. In other words, one has, so to speak, four particles to deal with. This analogy is obvious in the diagrams of Fig. 1.

According to the definition (3), the variables s, t, and u are the so-called total energy squared in the three channels appropriately defined. In the conventional approach,[3–5] these are simply the masses of the particles involved. For this reason, one may think, offhand, that the difference between the present approach and the conventional one is more or less a matter of terminology or interpretation, or the difference is, after all, very small even if it exists. This is correct in some sense, but there are rather surprising differences in some of the consequences of these two approaches.

[10] B. P. Roe, D. Sinclair, J. L. Brown, D. A. Glaser, J. A. Kadyk, and G. H. Trilling, Phys. Rev. Letters **7**, 346 (1961). According to Roe, a new determination of this branching ratio is now in progress based upon a larger number of events. According to Roe, however, the branching ratio is not expected to differ very much from 64% (private communication).

[11] M. Sugawara and T. Sakuma, following paper, Phys. Rev. **135**, B260 (1964).

In the case of the pionic decays of hyperons discussed in Sec. II, the usual heuristic argument implies that there should be no parity violation in the pole approximation in the conventional sense, while the pole approximation in the present approach is identical to the pole approximation due to Feldman, Matthews, and Salam.[8] If this pole approximation[8] is valid to some extent, one may say that the present approach is more useful than the conventional one in the case of the pionic decays of hyperons.

A very remarkable difference between these two approaches arises in the case of the leptonic decays of pions and kaons. It is shown in Sec. III that the invariant decay amplitude remains a constant, even off the energy-momentum shell. This means that one obtains no (nontrivial) dispersion relation in the present approach, in contrast with the various dispersion relations[3–5] proposed in the conventional approach.

The present approach differs also from the conventional one in interpreting the empirical fact that the ratio in (37) is almost unity. This can only be a pure accident according to the conventional approach, because these c's in (37) have complicated structures which are expressed by various dispersion relations.[3–5] On the other hand, the equality of these c's in (37) implies the universality of the weak interaction according to the present approach, because these c's in (37) are the genuine constants which can be identified as the weak coupling constants. According to Sec. III, these c's in (37) are also independent of whether the charged lepton is the electron or the μ meson. Therefore, the present approach is not only simpler but also more useful than the conventional one, as far as the leptonic decays of pions and kaons are concerned.

The above universality of the weak interaction is further extended in a subsequent paper[11] to the pionic decays of hyperons. The various constants a's which appear in the pole terms (19), (20), and (24) are first made dimensionless by introducing appropriate mass units. The above universality then implies that all these a's, after having been made dimensionless, are numerically the same as the c's in (37) and (38). It is found in the subsequent paper[11] that the available data

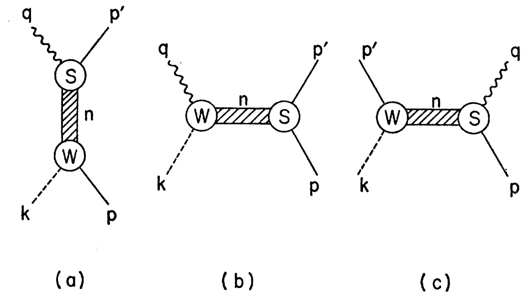

FIG. 1. Diagrams (a), (b), and (c) indicate those intermediate states, denoted by n, which give rise to the singularities with respect to s, t, and u, respectively. The vertices with W within circles stand for the weak vertices, while those with S are the strong ones. The dashed lines denoted by k indicate the fictitious particle mentioned in Sec. I of the text.

are actually consistent with the above universality as long as the pole approximation[8] is valid.

Therefore, it appears that the present approach is more useful than the conventional one as far as the pionic decays of hyperons and the leptonic decays of pions and kaons are concerned. However, the conventional approach has made a successful correlation[3–5] of the leptonic decays of pions with the β decay of nucleons. Therefore, it would not quite be until one works out the corresponding correlation according to the present approach that one can really tell which approach is more useful. The β decay of nucleons has not yet been fully investigated along the line of the present approach.

Besides the analyticity assumption, we assume in the present work some other basic assumptions. The basic assumptions are summarized at the end of the introduction. As is stated there, the existence of a localized weak Hamiltonian $H_W(x)$ appears to be a crucial assumption, because the present author does not know how to formulate the present approach without this assumption. The present approach can be formulated without some of the other basic assumptions and/or even if the weak boson exists. However, if the weak boson exists, almost all the details given in Secs. II and III have to be rederived because all the decays discussed there become of the second order with respect to $H_W(x)$.